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CÁLCULO DEL TENSOR DE RIEMANN
EN ESPACIO-TIEMPO DE SCHWARZSCHILD

Métrica de Schwarzschild

$$ds^2 = \frac{a-r}{r} c^2 dt^2 + \frac{r}{r-a} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

g_{00}	g_{11}	g_{22}	g_{33}
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$$g = \frac{2GM}{c^2}$$

Símbolos de Christoffel distintos de cero

$$x^0 = ct$$

$$x^1 = r$$

$$x^2 = \theta$$

$$x^3 = \phi$$

$\Gamma_{00}^1 = \frac{a(r-a)}{2r^3}$	$\Gamma_{01}^0 = \frac{a}{2r(r-a)}$	$\Gamma_{11}^1 = \frac{a}{2r(a-r)}$
$\Gamma_{12}^2 = \frac{1}{r}$	$\Gamma_{31}^3 = \frac{1}{r}$	$\Gamma_{22}^1 = a-r$
$\Gamma_{23}^3 = \cot\theta$	$\Gamma_{33}^2 = -\cos\theta \cdot \sin\theta$	$\Gamma_{33}^1 = (a-r)\sin^2\theta$

Método Simplificado, propuesto por Javier Garcia (Fórmula 16 A1.1):

$$[\partial_k \partial_l - \partial_l \partial_k] \vec{e}_j = R^i_{jkl} \vec{e}_i = R^0_{jkl} \vec{e}_0 + R^1_{jkl} \vec{e}_1 + R^2_{jkl} \vec{e}_2 + R^3_{jkl} \vec{e}_3$$

Al aplicar el operador $[\partial_k \partial_l - \partial_l \partial_k]$ a un vector de la base, se obtiene otro vector con 4 componentes y podemos deducir de golpe, con ese cálculo, 4 valores de R^i_{jkl} . El índice "i", muerto en el segundo miembro, se concreta una vez hecho el cálculo y comparar las componentes obtenidas con R^i_{jkl} .

Aplicaré el método eligiendo todas las combinaciones posibles de los valores (0, 1, 2, 3) para j, k, l, pero con dos restricciones:

- Evitar combinaciones en las que $K = l$, pues en esos casos resulta obvio que el operador será nulo y, por lo tanto, sabemos de antemano que son nulas todas las componentes del tipo: $(R^i_{jkl})_{l=k} = 0$
- No repetir aquellas combinaciones en las que K y l cambian de orden, pues también resulta obvio que el operador cambiará de signo y, por lo tanto, sabemos de antemano que cambiarán de signo las componentes: $(R^i_{jkl})_{l=k} = - (R^i_{jlk})_{l=k}$

Con estas restricciones, para cada uno de los cuatro valores de j = 0, 1, 2, 3 las combinaciones que consideraremos son:

$$(K=0 \ l=1)$$

$$(K=0 \ l=2) \quad (K=1 \ l=2)$$

$$(K=0 \ l=3) \quad (K=1 \ l=3) \quad (K=2 \ l=3)$$

$$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} 6 \times 4 \text{ valores de } j = 24$$

Por consiguiente tendré que realizar 24 cálculos, aplicando el método, que los voy enumerando en las páginas siguientes.

$$1) - j=0 \ k=0 \ l=1 \rightarrow [\partial_0 \partial_1 - \partial_1 \partial_0] \bar{e}_0 = R_{001}^i \cdot \bar{e}_i$$

$$\bullet \partial_0 (\partial_1 \bar{e}_0) = \partial_0 (\Gamma_{1,0}^m \bar{e}_m) = [sólo existe para m=0] = \partial_0 (\Gamma_{1,0}^0 \bar{e}_0) = (\partial_0 \Gamma_{1,0}^0) \bar{e}_0 + \Gamma_{1,0}^0 (\partial_0 \bar{e}_0) = \Gamma_{1,0}^0 \Gamma_{00}^m \bar{e}_m = [sólo existe para m=1] = \Gamma_{1,0}^0 \Gamma_{00}^1 \bar{e}_1$$

$$\bullet \partial_1 (\partial_0 \bar{e}_0) = \partial_1 (\Gamma_{00}^m \bar{e}_m) = [sólo existe para m=1] = \partial_1 (\Gamma_{00}^1 \bar{e}_1) = (\partial_1 \Gamma_{00}^1) \bar{e}_1 + \Gamma_{00}^1 (\partial_1 \bar{e}_1) = (\partial_1 \Gamma_{00}^1) \bar{e}_1 + \Gamma_{00}^1 \Gamma_{11}^m \bar{e}_m = [sólo existe para m=1] = (\partial_1 \Gamma_{00}^1) \bar{e}_1 + \Gamma_{00}^1 \Gamma_{11}^1 \bar{e}_1$$

Restando y agrupando: $[\partial_0 \partial_1 - \partial_1 \partial_0] \bar{e}_0 = 0 \bar{e}_0 + (\Gamma_{1,0}^0 \Gamma_{00}^1 - \partial_1 \Gamma_{00}^1 - \Gamma_{00}^1 \Gamma_{11}^1) \bar{e}_1 + 0 \bar{e}_2 + 0 \bar{e}_3$

Comparando:
con la expresión inicial

$$R_{001}^0 = 0 \quad R_{001}^1 = 0 \quad R_{001}^3 = 0$$

$$\begin{aligned} R_{001}^1 &= -\partial_1 \Gamma_{00}^1 + \Gamma_{1,0}^0 \Gamma_{00}^1 - \Gamma_{00}^1 \Gamma_{11}^1 = \\ &= -\partial_1 \frac{a(r-a)}{2r^3} + \frac{a}{2r(r-a)} \frac{a(r-a)}{2r^3} - \frac{a(r-a)}{2r^3} \frac{a}{2r(a-r)} = \\ &= \frac{-a^2 r^3 + 6r^2 a(r-a)}{4r^6} + \frac{a^2}{2r^4} = \dots \rightarrow R_{001}^1 = \frac{a}{r^4} (r-a) \end{aligned}$$

$$2) - j=0 \ k=0 \ l=2 \rightarrow [\partial_0 \partial_2 - \partial_2 \partial_0] \bar{e}_0 = R_{002}^i \cdot \bar{e}_i (*)$$

$$\bullet \partial_0 (\partial_2 \bar{e}_0) = \partial_0 \Gamma_{2,0}^m \bar{e}_m = [Todo \Gamma_{2,0}^m = 0] = 0$$

$$\bullet \partial_2 (\partial_0 \bar{e}_0) = \partial_2 (\Gamma_{00}^m \bar{e}_m) = [existe para m=1] = \partial_2 (\Gamma_{00}^1 \bar{e}_1) = (\partial_2 \Gamma_{00}^1) \bar{e}_1 + \Gamma_{00}^1 (\partial_2 \bar{e}_1) = \Gamma_{00}^1 \Gamma_{21}^m \bar{e}_m = [existe para m=2] = \Gamma_{00}^1 \Gamma_{21}^2 \bar{e}_2$$

Restando y agrupando: $[\partial_0 \partial_2 - \partial_2 \partial_0] \bar{e}_0 = 0 \bar{e}_0 + 0 \bar{e}_1 - \Gamma_{00}^1 \Gamma_{21}^2 \bar{e}_2 + 0 \bar{e}_3$

Comparando con (*) tenemos que:

$$R_{002}^0 = 0 \quad R_{002}^1 = 0 \quad R_{002}^3 = 0$$

$$R_{002}^2 = -\Gamma_{00}^1 \Gamma_{21}^2 = -\frac{a(r-a)}{2r^3} \cdot \frac{1}{r} \rightarrow R_{002}^2 = \frac{a(a-r)}{2r^4}$$

$$3) - \underbrace{j=0 \quad k=0 \quad l=3}_{\rightarrow} \Rightarrow [\partial_0 \partial_3 - \partial_3 \partial_0] \bar{e}_0 = R^i_{003} \bar{e}_i \quad (*)$$

$$\bullet \partial_0 (\partial_3 \bar{e}_0) = \partial_0 (\Gamma_{30}^m \bar{e}_m) = [\text{Todos } \Gamma_{30}^m = 0] = 0$$

$$\bullet \partial_3 (\partial_0 \bar{e}_0) = \partial_3 (\Gamma_{00}^m \bar{e}_m) = [m=1] = \partial_3 (\Gamma_{00}^1 \bar{e}_1) = (\partial_3 \Gamma_{00}^1) \bar{e}_1 + \Gamma_{00}^1 (\partial_3 \bar{e}_1) = \\ = \Gamma_{00}^1 \Gamma_{31}^m \bar{e}_m = [m=3] = \Gamma_{00}^1 \Gamma_{31}^3 \bar{e}_3$$

Restando:
ordenando: $[\partial_0 \partial_3 - \partial_3 \partial_0] \bar{e}_0 = 0 \bar{e}_0 + 0 \bar{e}_1 + 0 \bar{e}_2 - \Gamma_{00}^1 \Gamma_{31}^3 \bar{e}_3$

Comparando com (*) : $R^0_{003} = 0 \quad R^1_{003} = 0 \quad R^2_{003} = 0$

$$R^3_{003} = - \Gamma_{00}^1 \Gamma_{31}^3 = - \frac{\alpha(r-a)}{2r^3} \cdot \frac{1}{r} \Rightarrow R^3_{003} = \frac{\alpha(a-r)}{2r^4}$$

$$4) - \underbrace{j=0 \quad k=1 \quad l=2}_{\rightarrow} \Rightarrow [\partial_1 \partial_2 - \partial_2 \partial_1] \bar{e}_0 = R^i_{012} \bar{e}_i \quad (*)$$

$$\bullet \partial_1 (\partial_2 \bar{e}_0) = \partial_1 (\Gamma_{20}^m \bar{e}_m) = [\text{Todos } \Gamma_{20}^m = 0] = 0$$

$$\bullet \partial_2 (\partial_1 \bar{e}_0) = \partial_2 (\Gamma_{10}^m \bar{e}_m) = [m=0] = \partial_2 (\Gamma_{10}^0 \bar{e}_0) = (\partial_2 \Gamma_{10}^0) \bar{e}_0 + \Gamma_{10}^0 (\partial_2 \bar{e}_0) = \\ = \Gamma_{10}^0 \Gamma_{20}^m \bar{e}_m = [\text{Todos } \Gamma_{20}^m = 0] = 0$$

Restando:
 $[\partial_1 \partial_2 - \partial_2 \partial_1] \bar{e}_0 = 0 \bar{e}_0 + 0 \bar{e}_1 + 0 \bar{e}_2 + 0 \bar{e}_3$

Comparando com (*) : $R^0_{012} = 0 \quad R^1_{012} = 0 \quad R^2_{012} = 0 \quad R^3_{012} = 0$

$$5) - \underbrace{j=0 \quad k=1 \quad l=3}_{\rightarrow} \Rightarrow [\partial_1 \partial_3 - \partial_3 \partial_1] \bar{e}_0 = R^i_{013} \bar{e}_i \quad (*)$$

$$\bullet \partial_1 (\partial_3 \bar{e}_0) = \partial_1 (\Gamma_{30}^m \bar{e}_m) = [\text{Todos } \Gamma_{30}^m = 0] = 0$$

$$\bullet \partial_3 (\partial_1 \bar{e}_0) = \partial_3 (\Gamma_{10}^m \bar{e}_m) = [m=0] = \partial_3 (\Gamma_{10}^0 \bar{e}_0) = (\partial_3 \Gamma_{10}^0) \bar{e}_0 + \Gamma_{10}^0 (\partial_3 \bar{e}_0) = \\ = \Gamma_{10}^0 \Gamma_{30}^m \bar{e}_m = [\text{Todos } \Gamma_{30}^m = 0] = 0$$

Restando:
 $[\partial_1 \partial_3 - \partial_3 \partial_1] \bar{e}_0 = 0 \bar{e}_0 + 0 \bar{e}_1 + 0 \bar{e}_2 + 0 \bar{e}_3$

Comparando com (*) : $R^0_{013} = 0 \quad R^1_{013} = 0 \quad R^2_{013} = 0 \quad R^3_{013} = 0$

$$6) - j=0 \ k=2 \ l=3 \Rightarrow [\partial_2 \partial_3 - \partial_3 \partial_2] \bar{e}_0 = R^i_{023} \bar{e}_i$$

$$\bullet \partial_2 (\partial_3 \bar{e}_0) = \partial_2 (\Gamma_{30}^m \bar{e}_m) = [Todos \Gamma_{30}^m = 0] = 0$$

$$\bullet \partial_3 (\partial_2 \bar{e}_0) = \partial_3 (\Gamma_{20}^m \bar{e}_m) = [Todos \Gamma_{20}^m = 0] = 0$$

Restando queda: $[\partial_2 \partial_3 - \partial_3 \partial_2] \bar{e}_0 = 0 \bar{e}_0 + 0 \bar{e}_1 + 0 \bar{e}_2 + 0 \bar{e}_3$

Comparando $\rightarrow R^0_{023} = 0 \quad R^1_{023} = 0 \quad R^2_{023} = 0 \quad R^3_{023} = 0$

$$7) - j=1 \ k=0 \ l=1 \rightarrow [\partial_0 \partial_1 - \partial_1 \partial_0] \bar{e}_1 = R^i_{101} \bar{e}_i$$

$$\bullet \partial_0 (\partial_1 \bar{e}_1) = \partial_0 (\Gamma_{11}^m \bar{e}_m) = [m=1] = \partial_0 (\Gamma_{11}^1 \bar{e}_1) = (\partial_0 \Gamma_{11}^1) \bar{e}_1 + \Gamma_{11}^1 (\partial_0 \bar{e}_1) = \\ = \Gamma_{11}^1 \Gamma_{01}^m \bar{e}_m = [m=0] = \Gamma_{11}^1 \Gamma_{01}^0 \bar{e}_0$$

$$\bullet \partial_1 (\partial_0 \bar{e}_1) = \partial_1 (\Gamma_{01}^m \bar{e}_m) = [m=0] = \partial_1 (\Gamma_{01}^0 \bar{e}_0) = (\partial_1 \Gamma_{01}^0) \bar{e}_0 + \Gamma_{01}^0 (\partial_1 \bar{e}_0) = \\ = (\partial_1 \Gamma_{01}^0) \bar{e}_0 + \Gamma_{01}^0 (\Gamma_{10}^m \bar{e}_m) = [m=0] = (\partial_1 \Gamma_{01}^0) \bar{e}_0 + \Gamma_{01}^0 \cdot \Gamma_{10}^0 \bar{e}_0$$

Restando se obtiene: $(\partial_0 \partial_1 - \partial_1 \partial_0) \bar{e}_1 = (\Gamma_{11}^1 \Gamma_{01}^0 - \partial_1 \Gamma_{01}^0 - \Gamma_{01}^0 \Gamma_{10}^0) \bar{e}_0 + 0 \bar{e}_1 + 0 \bar{e}_2 + 0 \bar{e}_3$

Comparando $\rightarrow R^0_{101} = (-\partial_1 \Gamma_{01}^0 + \Gamma_{11}^1 \cdot \Gamma_{01}^0 - \Gamma_{01}^0 \Gamma_{10}^0) = \\ = -\partial_1 \left[\frac{a}{2r(r-a)} \right] + \frac{a}{2r(r-a)} \cdot \frac{a}{2r(r-a)} - \left[\frac{a}{2r(r-a)} \right]^2 = \frac{a(4r-2a)}{4r^2(r-a)^2} = \frac{2a^2}{4r^2(r-a)^2} =$

$$R^1_{101} = 0 \quad R^2_{101} = 0 \quad R^3_{101} = 0 \quad R^0_{101} = \frac{a}{r^2(r-a)}$$

$$8) - j=1 \ k=0 \ l=2 \rightarrow [\partial_0 \partial_2 - \partial_2 \partial_0] \bar{e}_1 = R^i_{102} \bar{e}_i$$

$$\bullet \partial_0 (\partial_2 \bar{e}_1) = \partial_0 (\Gamma_{21}^m \bar{e}_m) = [m=2] = \partial_0 (\Gamma_{21}^2 \bar{e}_2) = (\partial_0 \Gamma_{21}^2) \bar{e}_2 + \Gamma_{21}^2 (\partial_0 \bar{e}_2) = \\ = \Gamma_{21}^2 \Gamma_{02}^m \bar{e}_m = [Todos \Gamma_{02}^m = 0] = 0$$

$$\bullet \partial_2 (\partial_0 \bar{e}_1) = \partial_2 (\Gamma_{01}^m \bar{e}_m) = [m=0] = \partial_2 (\Gamma_{01}^0 \bar{e}_0) = (\partial_2 \Gamma_{01}^0) \bar{e}_0 + \Gamma_{01}^0 (\partial_2 \bar{e}_0) = \\ = \Gamma_{01}^0 \Gamma_{20}^m \bar{e}_m = [Todos \Gamma_{20}^m = 0] = 0$$

Restando queda: $(\partial_0 \partial_2 - \partial_2 \partial_0) \bar{e}_1 = 0 \bar{e}_0 + 0 \bar{e}_1 + 0 \bar{e}_2 + 0 \bar{e}_3$

Resulta que: $R^0_{102} = 0 \quad R^1_{102} = 0 \quad R^2_{102} = 0 \quad R^3_{102} = 0$

$$9) - \underbrace{j=1 \ k=0 \ l=3}_{\Rightarrow} \underbrace{[\partial_0 \partial_3 - \partial_3 \partial_0] \vec{e}_1 = R_{103}^i \vec{e}_i}_{0}$$

$$\bullet \partial_0(\partial_3 \vec{e}_1) = \partial_0(\Gamma_{31}^m \vec{e}_m) = [m=3] = \partial_0 \Gamma_{31}^3 \vec{e}_3 = (\partial_0 \Gamma_{31}^3) \vec{e}_3 + \Gamma_{31}^3 (\partial_0 \vec{e}_3) = \\ = \Gamma_{31}^3 \Gamma_{03}^m \vec{e}_m = [\text{Todos } \Gamma_{03}^m = 0] = 0$$

$$\bullet \partial_3(\partial_0 \vec{e}_1) = \partial_3(\Gamma_{01}^m \vec{e}_m) = [m=0] = \partial_3(\Gamma_{01}^0 \vec{e}_0) = (\partial_3 \Gamma_{01}^0) \vec{e}_0 + \Gamma_{01}^0 (\partial_3 \vec{e}_0) = \\ = \Gamma_{01}^0 \Gamma_{30}^m \vec{e}_m = [\text{Todos } \Gamma_{30}^m = 0] = 0$$

$$\text{Restando: } [\partial_0 \partial_3 - \partial_3 \partial_0] \vec{e}_1 = 0 \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + 0 \vec{e}_3$$

$$\text{dicho resultado: } \boxed{R_{103}^0 = 0} \quad \boxed{R_{103}^1 = 0} \quad \boxed{R_{103}^2 = 0} \quad \boxed{R_{103}^3 = 0}$$

$$10) - \underbrace{j=1 \ k=1 \ l=2}_{\Rightarrow} \underbrace{[\partial_1 \partial_2 - \partial_2 \partial_1] \vec{e}_1 = R_{112}^i \vec{e}_i}_{0}$$

$$\bullet \partial_1(\partial_2 \vec{e}_1) = \partial_1(\Gamma_{21}^m \vec{e}_m) = [m=2] = \partial_1(\Gamma_{21}^2 \vec{e}_2) = (\partial_1 \Gamma_{21}^2) \vec{e}_2 + \Gamma_{21}^2 (\partial_1 \vec{e}_2) = \\ = (\partial_1 \Gamma_{21}^2) \vec{e}_2 + \Gamma_{21}^2 \Gamma_{12}^m \vec{e}_m = [m=2] = (\partial_1 \Gamma_{21}^2) \vec{e}_2 + \Gamma_{21}^2 \Gamma_{12}^2 \vec{e}_2$$

$$\bullet \partial_2(\partial_1 \vec{e}_1) = \partial_2(\Gamma_{11}^m \vec{e}_m) = [m=1] = \partial_2(\Gamma_{11}^1 \vec{e}_1) = (\partial_2 \Gamma_{11}^1) \vec{e}_1 + \Gamma_{11}^1 (\partial_2 \vec{e}_1) = \\ = \Gamma_{11}^1 \Gamma_{21}^m \vec{e}_m = [m=2] = \Gamma_{11}^1 \Gamma_{21}^2 \vec{e}_2$$

$$\text{Restando: } [\partial_1 \partial_2 - \partial_2 \partial_1] \vec{e}_1 = 0 \vec{e}_0 + 0 \vec{e}_1 + (\partial_1 \Gamma_{21}^2 + \Gamma_{21}^2 \Gamma_{12}^2 - \Gamma_{11}^1 \Gamma_{21}^2) \vec{e}_2 + 0 \vec{e}_3$$

$$\text{Resultado: } R_{112}^2 = \partial_1 \Gamma_{21}^2 + \Gamma_{21}^2 \Gamma_{12}^2 - \Gamma_{11}^1 \Gamma_{21}^2 = \partial_r \left(\frac{1}{r} \right) + \left(\frac{1}{r} \right)^2 - \frac{a}{2r(a-r)} \cdot \frac{1}{r} =$$

$$\boxed{R_{112}^0 = 0} \quad \boxed{R_{112}^1 = 0} \quad \boxed{R_{112}^3 = 0} \quad \rightarrow -\frac{1}{r^2} + \frac{1}{r^2} - \frac{a}{2r^2(a-r)} \rightarrow \boxed{R_{112}^2 = \frac{a}{2r^2(r-a)}}$$

$$11) - \underbrace{j=1 \ k=1 \ l=3}_{\Rightarrow} \underbrace{[\partial_1 \partial_3 - \partial_3 \partial_1] \vec{e}_1 = R_{113}^i \vec{e}_i}_{0}$$

$$\bullet \partial_1(\partial_3 \vec{e}_1) = \partial_1(\Gamma_{31}^m \vec{e}_m) = [m=3] = \partial_1(\Gamma_{31}^3 \vec{e}_3) = (\partial_1 \Gamma_{31}^3) \vec{e}_3 + \Gamma_{31}^3 (\partial_1 \vec{e}_3) = \\ = (\partial_1 \Gamma_{31}^3) \vec{e}_3 + \Gamma_{31}^3 \Gamma_{13}^m \vec{e}_m = [m=3] = (\partial_1 \Gamma_{31}^3) \vec{e}_3 + \Gamma_{31}^3 \Gamma_{13}^3 \vec{e}_3$$

$$\bullet \partial_3(\partial_1 \vec{e}_1) = \partial_3(\Gamma_{11}^m \vec{e}_m) = [m=1] = \partial_3(\Gamma_{11}^1 \vec{e}_1) = (\partial_3 \Gamma_{11}^1) \vec{e}_1 + \Gamma_{11}^1 (\partial_3 \vec{e}_1) = \\ = \Gamma_{11}^1 \Gamma_{31}^m \vec{e}_m = [m=3] = \Gamma_{11}^1 \Gamma_{31}^3 \vec{e}_3$$

$$\text{Restando: } [\partial_1 \partial_3 - \partial_3 \partial_1] \vec{e}_1 = 0 \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + (\partial_1 \Gamma_{31}^3 + \Gamma_{31}^3 \Gamma_{13}^3 - \Gamma_{11}^1 \Gamma_{31}^3) \vec{e}_3$$

$$\text{Resultado: } \boxed{R_{113}^0 = 0} \quad \boxed{R_{113}^1 = 0} \quad \boxed{R_{113}^2 = 0}$$

$$R_{113}^3 = \partial_1 \Gamma_{31}^3 + \Gamma_{31}^3 \Gamma_{13}^3 - \Gamma_{11}^1 \Gamma_{31}^3 = \partial_r \left(\frac{1}{r} \right) + \left(\frac{1}{r} \right)^2 - \frac{a}{2r(a-r)} \cdot \frac{1}{r} =$$

$$= -\frac{1}{r^2} + \frac{1}{r^2} - \frac{a}{2r^2(a-r)} \rightarrow \boxed{R_{113}^3 = \frac{a}{2r^2(r-a)}}$$

$$12) - \underbrace{j=1 \ k=2 \ l=3}_{\rightarrow} [\partial_2 \partial_3 - \partial_3 \partial_2] \vec{e}_1 = R^i_{123} \vec{e}_i$$

$$\bullet \partial_2(\partial_3 \vec{e}_1) = \partial_2(\Gamma_{31}^m \vec{e}_m) = [m=3] = \partial_2(\Gamma_{31}^3 \vec{e}_3) = (\partial_2 \Gamma_{31}^3) \vec{e}_3 + \Gamma_{31}^3 (\partial_2 \vec{e}_3) = \Gamma_{31}^3 \Gamma_{23}^m \vec{e}_m = [m=3] = \Gamma_{31}^3 \Gamma_{23}^3 \vec{e}_3$$

$$\bullet \partial_3(\partial_2 \vec{e}_1) = \partial_3(\Gamma_{21}^m \vec{e}_m) = [m=2] = \partial_3(\Gamma_{21}^2 \vec{e}_2) = (\partial_3 \Gamma_{21}^2) \vec{e}_2 + \Gamma_{21}^2 (\partial_3 \vec{e}_2) = \Gamma_{21}^2 \Gamma_{32}^m \vec{e}_m = [m=3] = \Gamma_{21}^2 \Gamma_{32}^3 \vec{e}_3$$

Restando queda: $[\partial_2 \partial_3 - \partial_3 \partial_2] \vec{e}_1 = 0 \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + (\Gamma_{31}^3 \Gamma_{23}^3 - \Gamma_{21}^2 \Gamma_{32}^3) \vec{e}_3$

Por lo tanto: $R^0_{123} = 0$ $R^1_{123} = 0$ $R^2_{123} = 0$ $R^3_{123} = 0$

$$R^3_{123} = \Gamma_{31}^3 \Gamma_{23}^3 - \Gamma_{21}^2 \Gamma_{32}^3 = \frac{1}{r} \cot \theta - \frac{1}{r} \cot \theta$$

$$13) - \underbrace{j=2 \ k=0 \ l=1}_{\rightarrow} [\partial_0 \partial_1 - \partial_1 \partial_0] \vec{e}_2 = R^i_{201} \vec{e}_i$$

$$\bullet \partial_0(\partial_1 \vec{e}_2) = \partial_0(\Gamma_{12}^m \vec{e}_m) = [m=2] = \partial_0(\Gamma_{12}^2 \vec{e}_2) = (\partial_0 \Gamma_{12}^2) \vec{e}_2 + \Gamma_{12}^2 (\partial_0 \vec{e}_2) = \Gamma_{12}^2 \Gamma_{02}^m \vec{e}_m = [Todo \Gamma_{02}^m = 0] = 0$$

$$\bullet \partial_1(\partial_0 \vec{e}_2) = \partial_1(\Gamma_{02}^m \vec{e}_m) = [Todo \Gamma_{02}^m = 0] = 0$$

Queda: $[\partial_0 \partial_1 - \partial_1 \partial_0] \vec{e}_2 = 0 \vec{e}_1 + 0 \vec{e}_2 + 0 \vec{e}_3 + 0 \vec{e}_4$

despues: $R^0_{201} = 0$ $R^1_{201} = 0$ $R^2_{201} = 0$ $R^3_{201} = 0$

$$14) - \underbrace{j=2 \ k=0 \ l=2}_{\rightarrow} [\partial_0 \partial_2 - \partial_2 \partial_0] \vec{e}_2 = R^i_{202} \vec{e}_i$$

$$\bullet \partial_0(\partial_2 \vec{e}_2) = \partial_0(\Gamma_{22}^m \vec{e}_m) = [m=1] = \partial_0(\Gamma_{22}^1 \vec{e}_1) = (\partial_0 \Gamma_{22}^1) \vec{e}_1 + \Gamma_{22}^1 (\partial_0 \vec{e}_1) = \Gamma_{22}^1 \Gamma_{01}^m \vec{e}_m = [m=0] = \Gamma_{22}^1 \Gamma_{01}^0 \vec{e}_0$$

$$\bullet \partial_2(\partial_0 \vec{e}_2) = \partial_2(\Gamma_{02}^m \vec{e}_m) = [Todo \Gamma_{02}^m = 0] = 0$$

Restando queda: $[\partial_0 \partial_2 - \partial_2 \partial_0] \vec{e}_2 = (\Gamma_{22}^1 \Gamma_{01}^0) \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + 0 \vec{e}_3$

despues resulta: $R^0_{202} = \Gamma_{22}^1 \Gamma_{01}^0 = (a-r) \frac{a}{2r(r-a)} \rightarrow R^0_{202} = -\frac{a}{2r}$

$$R^1_{202} = 0$$

$$R^2_{202} = 0$$

$$R^3_{202} = 0$$

$$15) - \underbrace{j=2 \ k=0 \ l=3}_{\rightarrow (\partial_0 \partial_3 - \partial_3 \partial_0) \bar{e}_2 = R^i_{203} \bar{e}_i}$$

$$\bullet \partial_0(\partial_3 \bar{e}_2) = \partial_0(\Gamma_{32}^m \bar{e}_m) = [m=3] = \partial_0(\Gamma_{32}^3 \bar{e}_3) = (\partial_0 \Gamma_{32}^3) \bar{e}_3 + \Gamma_{32}^3 (\partial_0 \bar{e}_3) = \\ = \Gamma_{32}^3 \Gamma_{03}^m \bar{e}_m = [Todas \ \Gamma_{03}^m = 0] = 0$$

$$\bullet \partial_3(\partial_0 \bar{e}_2) = \partial_3(\Gamma_{02}^m \bar{e}_m) = [Todas \ \Gamma_{02}^m = 0] = 0$$

Restamos y queda: $(\partial_0 \partial_3 - \partial_3 \partial_0) \bar{e}_2 = 0 \bar{e}_0 + 0 \bar{e}_1 + 0 \bar{e}_2 + 0 \bar{e}_3$

despues: $R^0_{203} = 0 \quad R^1_{203} = 0 \quad R^2_{203} = 0 \quad R^3_{203} = 0$

$$16) - \underbrace{j=2 \ k=1 \ l=2}_{\rightarrow (\partial_1 \partial_2 - \partial_2 \partial_1) \bar{e}_2 = R^i_{212} \bar{e}_i}$$

$$\bullet \partial_1(\partial_2 \bar{e}_2) = \partial_1(\Gamma_{22}^m \bar{e}_m) = [m=1] = \partial_1(\Gamma_{22}^1 \bar{e}_1) = (\partial_1 \Gamma_{22}^1) \bar{e}_1 + \Gamma_{22}^1 (\partial_1 \bar{e}_1) = \\ = (\partial_1 \Gamma_{22}^1) \bar{e}_1 + \Gamma_{22}^1 (\Gamma_{11}^m \bar{e}_m) = [m=1] = (\partial_1 \Gamma_{22}^1) \bar{e}_1 + \Gamma_{22}^1 \Gamma_{11}^1 \bar{e}_1$$

$$\bullet \partial_2(\partial_1 \bar{e}_2) = \partial_2(\Gamma_{12}^m \bar{e}_m) = [m=2] = \partial_2(\Gamma_{12}^2 \bar{e}_2) = (\partial_2 \Gamma_{12}^2) \bar{e}_2 + \Gamma_{12}^2 (\partial_2 \bar{e}_2) = \\ = \Gamma_{12}^2 \Gamma_{22}^m \bar{e}_m = [m=1] = \Gamma_{12}^2 \Gamma_{22}^1 \bar{e}_1$$

Se resta y queda: $[\partial_1 \partial_2 - \partial_2 \partial_1] \bar{e}_2 = 0 \bar{e}_0 + (\partial_1 \Gamma_{22}^1 + \Gamma_{22}^1 \Gamma_{11}^1 - \Gamma_{12}^2 \Gamma_{22}^1) \bar{e}_1 + 0 \bar{e}_2 + 0 \bar{e}_3$

despues: $R^1_{212} = (\partial_1 \Gamma_{22}^1 + \Gamma_{22}^1 \Gamma_{11}^1 - \Gamma_{12}^2 \Gamma_{22}^1) = \partial_r(a-r) + (a-r) \frac{\alpha}{2r(a-r)} - \frac{1}{r}(a-r) = \\ = -1 + \frac{\alpha}{2r} - \frac{\alpha}{r} + 1 \Rightarrow R^1_{212} = -\frac{\alpha}{2r}$

$$R^0_{212} = 0$$

$$R^2_{212} = 0$$

$$R^3_{212} = 0$$

$$17) - \underbrace{j=2 \ k=1 \ l=3}_{\rightarrow (\partial_1 \partial_3 - \partial_3 \partial_1) \bar{e}_2 = R^i_{213} \bar{e}_i}$$

$$\bullet \partial_1(\partial_3 \bar{e}_2) = \partial_1(\Gamma_{32}^m \bar{e}_m) = [m=3] = \partial_1(\Gamma_{32}^3 \bar{e}_3) = (\partial_1 \Gamma_{32}^3) \bar{e}_3 + \Gamma_{32}^3 (\partial_1 \bar{e}_3) = \\ = \Gamma_{32}^3 \Gamma_{13}^m \bar{e}_m = [m=3] = \Gamma_{32}^3 \Gamma_{13}^3 \bar{e}_3$$

$$\bullet \partial_3(\partial_1 \bar{e}_2) = \partial_3(\Gamma_{12}^m \bar{e}_m) = [m=2] = \partial_3(\Gamma_{12}^2 \bar{e}_2) = (\partial_3 \Gamma_{12}^2) \bar{e}_2 + \Gamma_{12}^2 (\partial_3 \bar{e}_2) = \\ = \Gamma_{12}^2 \Gamma_{32}^m \bar{e}_m = [m=3] = \Gamma_{12}^2 \Gamma_{32}^3 \bar{e}_3$$

Al restar queda: $[\partial_1 \partial_3 - \partial_3 \partial_1] \bar{e}_2 = 0 \bar{e}_0 + 0 \bar{e}_1 + 0 \bar{e}_2 + (\Gamma_{32}^3 \Gamma_{13}^3 - \Gamma_{12}^2 \Gamma_{32}^3) \bar{e}_3$

despues $R^3_{213} = \Gamma_{32}^3 \Gamma_{13}^3 - \Gamma_{12}^2 \Gamma_{32}^3 = (\cot \theta) \frac{1}{r} - \frac{1}{r} (\cot \theta) \Rightarrow R^3_{213} = 0$

$$R^0_{213} = 0$$

$$R^1_{213} = 0$$

$$R^2_{213} = 0$$

$$18) - \underline{j=2 \ k=2 \ l=3} \rightarrow (\partial_2 \partial_3 - \partial_3 \partial_2) \vec{e}_2 = R_{223}^i \vec{e}_i$$

$$\begin{aligned} \bullet \partial_2(\partial_3 \vec{e}_2) &= \partial_2(\Gamma_{32}^m \vec{e}_m) = [m=3] = \partial_2(\Gamma_{32}^3 \vec{e}_3) = (\partial_2 \Gamma_{32}^3) \vec{e}_3 + \Gamma_{32}^3 (\partial_2 \vec{e}_3) = \\ &= (\partial_2 \Gamma_{32}^3) \vec{e}_3 + \Gamma_{32}^3 \Gamma_{23}^m \vec{e}_m = [m=3] = (\partial_2 \Gamma_{32}^3) \vec{e}_3 + \Gamma_{32}^3 \Gamma_{23}^3 \vec{e}_3 \\ \bullet \partial_3(\partial_2 \vec{e}_2) &= \partial_3(\Gamma_{22}^m \vec{e}_m) = [m=1] = \partial_3(\Gamma_{22}^1 \vec{e}_1) = (\partial_3 \Gamma_{22}^1) \vec{e}_1 + \Gamma_{22}^1 (\partial_3 \vec{e}_1) = \\ &= \Gamma_{22}^1 \Gamma_{31}^m \vec{e}_m = [m=3] = \Gamma_{22}^1 \Gamma_{31}^3 \vec{e}_3 \end{aligned}$$

$$\text{Tras restar: } (\partial_2 \partial_3 - \partial_3 \partial_2) \vec{e}_2 = 0 \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + (\partial_2 \Gamma_{32}^3 + \Gamma_{32}^3 \Gamma_{23}^3 - \Gamma_{22}^1 \Gamma_{31}^3) \vec{e}_3$$

$$\begin{aligned} \text{Resulta: } R_{223}^3 &= \partial_2 \Gamma_{32}^3 + \Gamma_{32}^3 \Gamma_{23}^3 - \Gamma_{22}^1 \Gamma_{31}^3 = \partial_2 \cot \theta + (\cot \theta)^2 - (a-r) \frac{1}{r} = \\ &= -\frac{1}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} - \frac{a}{r} + 1 = -\frac{1-\cos^2 \theta}{\sin^2 \theta} - \frac{a}{r} + 1 \rightarrow R_{223}^3 = -\frac{a}{r} \end{aligned}$$

$$R_{223}^0 = 0 \quad R_{223}^1 = 0 \quad R_{223}^2 = 0$$

$$19) - \underline{j=3 \ k=0 \ l=1} \rightarrow (\partial_0 \partial_1 - \partial_1 \partial_0) \vec{e}_3 = R_{301}^i \vec{e}_i$$

$$\begin{aligned} \bullet \partial_0(\partial_1 \vec{e}_3) &= \partial_0(\Gamma_{13}^m \vec{e}_m) = [m=3] = \partial_0(\Gamma_{13}^3 \vec{e}_3) = (\partial_0 \Gamma_{13}^3) \vec{e}_3 + \Gamma_{13}^3 (\partial_0 \vec{e}_3) = \\ &= \Gamma_{13}^3 \Gamma_{03}^m \vec{e}_m = [\text{Todos } \Gamma_{03}^m = 0] = 0 \end{aligned}$$

$$\bullet \partial_1(\partial_0 \vec{e}_3) = \partial_1(\Gamma_{03}^m \vec{e}_m) = [\text{Todos } \Gamma_{03}^m = 0] = 0$$

$$\text{Restamos: } (\partial_0 \partial_1 - \partial_1 \partial_0) \vec{e}_3 = 0 \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + 0 \vec{e}_3$$

$$\text{dijo que: } R_{301}^0 = 0 \quad R_{301}^1 = 0 \quad R_{301}^2 = 0 \quad R_{301}^3 = 0$$

$$20) - \underline{j=3 \ k=0 \ l=2} \rightarrow (\partial_0 \partial_2 - \partial_2 \partial_0) \vec{e}_3 = R_{302}^i \vec{e}_i$$

$$\begin{aligned} \bullet \partial_0(\partial_2 \vec{e}_3) &= \partial_0(\Gamma_{23}^m \vec{e}_m) = [m=3] = \partial_0(\Gamma_{23}^3 \vec{e}_3) = (\partial_0 \Gamma_{23}^3) \vec{e}_3 + \Gamma_{23}^3 (\partial_0 \vec{e}_3) = \\ &= \Gamma_{23}^3 \Gamma_{03}^m \vec{e}_m = [\text{Todos } \Gamma_{03}^m = 0] = 0 \end{aligned}$$

$$\bullet \partial_2(\partial_0 \vec{e}_3) = \partial_2(\Gamma_{03}^m \vec{e}_m) = [\text{Todos } \Gamma_{03}^m = 0] = 0$$

$$\text{Restamos y queda: } (\partial_0 \partial_2 - \partial_2 \partial_0) \vec{e}_3 = 0 \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + 0 \vec{e}_3$$

dijo resultado:

$$R_{302}^0 = 0 \quad R_{302}^1 = 0 \quad R_{302}^2 = 0 \quad R_{302}^3 = 0$$

$$21) - j=3 \quad k=0 \quad l=3 \rightarrow (\partial_0 \partial_3 - \partial_3 \partial_0) \vec{e}_3 = R^i_{303} \vec{e}_i$$

$$\begin{aligned} \bullet \partial_0(\partial_3 \vec{e}_3) &= \partial_0(\Gamma_{33}^m \vec{e}_m) = [m=1, 2] = \partial_0(\Gamma_{33}^1 \vec{e}_1 + \Gamma_{33}^2 \vec{e}_2) = \\ &= (\partial_0 \Gamma_{33}^1) \vec{e}_1 + \Gamma_{33}^1 (\partial_0 \vec{e}_1) + (\partial_0 \Gamma_{33}^2) \vec{e}_2 + \Gamma_{33}^2 (\partial_0 \vec{e}_2) = \\ &= \Gamma_{33}^1 \Gamma_{01}^m \vec{e}_m + \Gamma_{33}^2 \Gamma_{02}^m \vec{e}_m = [m=0 \text{ y } \text{Todo } \Gamma_{02}^m = 0] = \Gamma_{33}^1 \Gamma_{01}^0 \vec{e}_0 \end{aligned}$$

$$\bullet \partial_3(\partial_0 \vec{e}_3) = \partial_3(\Gamma_{03}^m \vec{e}_m) = [\text{Todo } \Gamma_{03}^m = 0] = 0$$

Restando queda: $(\partial_0 \partial_3 - \partial_3 \partial_0) \vec{e}_3 = (\Gamma_{33}^1 \Gamma_{01}^0) \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + 0 \vec{e}_3$

Por lo tanto: $R^0_{303} = \Gamma_{33}^1 \Gamma_{01}^0 = (a-r) \sin^2 \theta \cdot \frac{a}{2r(r-a)} =$

$$\boxed{R^1_{303} = 0} \quad \boxed{R^2_{303} = 0} \quad \boxed{R^3_{303} = 0} \quad \Rightarrow \boxed{R^0_{303} = -\frac{a \sin^2 \theta}{2r}}$$

$$22) - j=3 \quad k=1 \quad l=2 \rightarrow (\partial_1 \partial_2 - \partial_2 \partial_1) \vec{e}_3 = R^i_{312} \vec{e}_i$$

$$\begin{aligned} \bullet \partial_1(\partial_2 \vec{e}_3) &= \partial_1(\Gamma_{23}^m \vec{e}_m) = [m=3] = \partial_1(\Gamma_{23}^3 \vec{e}_3) = (\partial_1 \Gamma_{23}^3) \vec{e}_3 + \Gamma_{23}^3 (\partial_1 \vec{e}_3) = \\ &= \Gamma_{23}^3 \Gamma_{13}^m \vec{e}_m = [m=3] = \Gamma_{23}^3 \Gamma_{13}^3 \vec{e}_3 \end{aligned}$$

$$\begin{aligned} \bullet \partial_2(\partial_1 \vec{e}_3) &= \partial_2(\Gamma_{13}^m \vec{e}_m) = [m=3] = \partial_2(\Gamma_{13}^3 \vec{e}_3) = (\partial_2 \Gamma_{13}^3) \vec{e}_3 + \Gamma_{13}^3 (\partial_2 \vec{e}_3) = \\ &= \Gamma_{13}^3 \Gamma_{23}^m \vec{e}_m = [m=3] = \Gamma_{13}^3 \Gamma_{23}^3 \vec{e}_3 \end{aligned}$$

Al restar queda: $(\partial_1 \partial_2 - \partial_2 \partial_1) \vec{e}_3 = 0 \vec{e}_0 + 0 \vec{e}_1 + 0 \vec{e}_2 + (\Gamma_{23}^3 \Gamma_{13}^3 - \Gamma_{13}^3 \Gamma_{23}^3) \vec{e}_3$

Por lo tanto, comparando componentes, queda

$$\begin{aligned} R^3_{312} &= (\Gamma_{23}^3 \Gamma_{13}^3 - \Gamma_{13}^3 \Gamma_{23}^3) = \\ &= (\cot \theta) \cdot \frac{1}{r} - \frac{1}{r} \cdot (\cot \theta) = 0 \end{aligned}$$

$$\boxed{R^0_{312} = 0} \quad \boxed{R^1_{312} = 0} \quad \boxed{R^2_{312} = 0} \quad \boxed{R^3_{312} = 0}$$

$$23) - \underbrace{j=3 \ k=1 \ l=3}_{\rightarrow (\partial_1 \partial_3 - \partial_3 \partial_1) \vec{e}_3 = R^i_{313} \vec{e}_i}$$

$$\begin{aligned} \bullet \partial_1(\partial_3 \vec{e}_3) &= \partial_1(\Gamma_{33}^m \vec{e}_m) = [m=1, 2] = \partial_1(\Gamma_{33}^1 \vec{e}_1 + \Gamma_{33}^2 \vec{e}_2) = \\ &= (\partial_1 \Gamma_{33}^1) \vec{e}_1 + \Gamma_{33}^1 (\partial_1 \vec{e}_1) + (\partial_1 \Gamma_{33}^2) \vec{e}_2 + \Gamma_{33}^2 (\partial_1 \vec{e}_2) = \\ &= (\partial_1 \Gamma_{33}^1) \vec{e}_1 + \Gamma_{33}^1 (\Gamma_{11}^m \vec{e}_m) + \Gamma_{33}^2 (\Gamma_{12}^m \vec{e}_n) = [m=1, n=2] = \\ &= (\partial_1 \Gamma_{33}^1) \vec{e}_1 + \Gamma_{33}^1 \Gamma_{11}^1 \vec{e}_1 + \Gamma_{33}^2 \Gamma_{12}^2 \vec{e}_2 \end{aligned}$$

$$\begin{aligned} \bullet \partial_3(\partial_1 \vec{e}_3) &= \partial_3(\Gamma_{13}^m \vec{e}_m) = [m=3] = \partial_3(\Gamma_{13}^3 \vec{e}_3) = (\partial_3 \Gamma_{13}^3) \vec{e}_3 + \Gamma_{13}^3 (\partial_3 \vec{e}_3) = \\ &= \Gamma_{13}^3 (\Gamma_{33}^m \vec{e}_m) = [m=1, 2] = \Gamma_{13}^3 (\Gamma_{33}^1 \vec{e}_1 + \Gamma_{33}^2 \vec{e}_2) = \Gamma_{13}^3 \Gamma_{33}^1 \vec{e}_1 + \Gamma_{13}^3 \Gamma_{33}^2 \vec{e}_2 \end{aligned}$$

Restamos, agrupamos por componentes, y queda:

$$(\partial_1 \partial_3 - \partial_3 \partial_1) \vec{e}_3 = 0 \vec{e}_0 + (\partial_1 \Gamma_{33}^1 + \Gamma_{33}^1 \Gamma_{11}^1 - \Gamma_{13}^3 \Gamma_{33}^1) \vec{e}_1 + (\Gamma_{33}^2 \Gamma_{12}^2 - \Gamma_{13}^3 \Gamma_{33}^2) \vec{e}_2 + 0 \vec{e}_3$$

$$\text{Resulta: } R^0_{313} = 0 \quad R^1_{313} = \partial_1 \Gamma_{33}^1 + \Gamma_{33}^1 \Gamma_{11}^1 - \Gamma_{13}^3 \Gamma_{33}^1 =$$

$$\begin{aligned} &= \partial_1[(a-r) \sin^2 \theta] + (a-r) \sin^2 \theta \frac{a}{2r(a-r)} - \frac{1}{r} (a-r) \sin^2 \theta = \\ &= -\sin^2 \theta + \frac{a \sin^2 \theta}{2r} - \frac{a \sin^2 \theta + r \sin^2 \theta}{r} = \rightarrow R^1_{313} = -\frac{a \sin^2 \theta}{2r} \end{aligned}$$

$$R^2_{313} = \Gamma_{33}^2 \Gamma_{12}^2 - \Gamma_{13}^3 \Gamma_{33}^2 = -\cos \theta \sin \theta \cdot \frac{1}{r} - \frac{1}{r} (-\cos \theta \sin \theta) \rightarrow R^2_{313} = 0 \quad R^3_{313} = 0$$

$$24) - \underbrace{j=3 \ k=2 \ l=3}_{\rightarrow (\partial_2 \partial_3 - \partial_3 \partial_2) \vec{e}_3 = R^i_{323} \vec{e}_i}$$

$$\begin{aligned} \bullet \partial_2(\partial_3 \vec{e}_3) &= \partial_2(\Gamma_{33}^m \vec{e}_m) = [m=1, 2] = \partial_2(\Gamma_{33}^1 \vec{e}_1 + \Gamma_{33}^2 \vec{e}_2) = \\ &= (\partial_2 \Gamma_{33}^1) \vec{e}_1 + \Gamma_{33}^1 (\partial_2 \vec{e}_1) + (\partial_2 \Gamma_{33}^2) \vec{e}_2 + \Gamma_{33}^2 (\partial_2 \vec{e}_2) = \\ &= (\partial_2 \Gamma_{33}^1) \vec{e}_1 + \Gamma_{33}^1 (\Gamma_{21}^m \vec{e}_m) + (\partial_2 \Gamma_{33}^2) \vec{e}_2 + \Gamma_{33}^2 (\Gamma_{22}^m \vec{e}_m) = [m=2, m=1] = \\ &= (\partial_2 \Gamma_{33}^1) \vec{e}_1 + \Gamma_{33}^1 \Gamma_{21}^2 \vec{e}_2 + (\partial_2 \Gamma_{33}^2) \vec{e}_2 + \Gamma_{33}^2 \Gamma_{22}^1 \vec{e}_1 \end{aligned}$$

$$\begin{aligned} \bullet \partial_3(\partial_2 \vec{e}_3) &= \partial_3(\Gamma_{23}^m \vec{e}_m) = [m=3] = \partial_3(\Gamma_{23}^3 \vec{e}_3) = (\partial_3 \Gamma_{23}^3) \vec{e}_3 + \Gamma_{23}^3 (\partial_3 \vec{e}_3) = \\ &= \Gamma_{23}^3 \Gamma_{33}^m \vec{e}_m = [m=1, 2] = \Gamma_{23}^3 (\Gamma_{33}^1 \vec{e}_1 + \Gamma_{33}^2 \vec{e}_2) = \\ &= \Gamma_{23}^3 \Gamma_{33}^1 \vec{e}_1 + \Gamma_{23}^3 \Gamma_{33}^2 \vec{e}_2 \end{aligned}$$

Restamos, agrupamos las componentes, y nos queda:

$$(\partial_2 \partial_3 - \partial_3 \partial_2) \vec{e}_3 = 0 \vec{e}_0 + (\partial_2 \Gamma_{33}^1 + \Gamma_{33}^1 \Gamma_{21}^2 - \Gamma_{23}^3 \Gamma_{33}^1) \vec{e}_1 + (\partial_2 \Gamma_{33}^2 + \Gamma_{33}^2 \Gamma_{22}^1 - \Gamma_{23}^3 \Gamma_{33}^2) \vec{e}_2 + 0 \vec{e}_3$$

Comparando con la expresión inicial (en función de los R^i_{323}) e igualando componentes resulta:

(continúa página siguiente)

$$R^0_{323} = 0$$

$$\begin{aligned} R^1_{323} &= \partial_2 \Gamma^1_{33} + \Gamma^2_{33} \Gamma^1_{22} - \Gamma^3_{23} \Gamma^1_{33} \\ &= \partial_\theta [(\alpha-r) \sin^2 \theta] + (-\cos \theta \sin \theta)(\alpha-r) - (\cot \theta)(\alpha-r) \sin^2 \theta \\ &= 2(\alpha-r) \sin \theta \cos \theta - (\alpha-r) \cos \theta \sin \theta - (\alpha-r) \cos \theta \sin \theta = 0 \end{aligned}$$

$$R^1_{323} = 0$$

$$\begin{aligned} R^2_{323} &= \partial_2 \Gamma^2_{33} + \Gamma^1_{33} \Gamma^2_{22} - \Gamma^3_{23} \Gamma^2_{33} = \\ &= \partial_\theta (-\cos \theta \sin \theta) + (\alpha-r) \sin^2 \theta \left(\frac{1}{r}\right) - \cot \theta \cdot (-\cos \theta \sin \theta) = \\ &= -(-\sin^2 \theta + \cos^2 \theta) + \frac{\alpha-r}{r} \sin^2 \theta + \cos^2 \theta = \\ &= \sin^2 \theta - \cos^2 \theta + \frac{\alpha-r}{r} \sin^2 \theta + \cos^2 \theta = \sin^2 \theta \left(1 + \frac{\alpha-r}{r}\right) = \sin^2 \theta \cdot \frac{\alpha}{r} \end{aligned}$$

$$R^2_{323} = \frac{\alpha \sin^2 \theta}{r}$$

$$R^3_{323} = 0$$



- Con estos últimos cuatro valores he concluido ¡por fin! los 24 cálculos que permiten deducir $24 \times 4 = 96$ componentes R^i_{jkl}
- Otras 96 componentes, las que se obtienen cambiando el orden de los dos últimos subíndices, están también determinadas, pues se cumple $R^i_{jkl} = -R^i_{jlk}$
- Además, todas aquellas componentes cuyos dos últimos subíndices sean iguales ($k = l$), sabemos que tienen que ser nulas. $R^i_{jll} = 0$. El número de estas componentes (no contempladas en los cálculos anteriores) será: $4 \times 4 \times 4 = 64$
- Por consiguiente, en total tenemos determinados los valores de: $96 + 96 + 64 = 256$ componentes del tensor de Riemann en el espacio-tiempo de Schwarzschild de 4 dimensiones, (Para n dimensiones: N^2 componentes = $M^4 = 4^4 = 256$)

RECAPITULACIÓN

Componentes NO NULAS del Tensor de Riemann

En la tabla se exponen todos las componentes (de las 256 que existen) que son distintas de cero

Se indica el Nº del cálculo en que se deduce

Sabemos que $a = \frac{2GM}{c^2}$

$\text{1)} R_{101}^0 = \frac{a}{r^2(r-a)}$	$\text{14)} R_{202}^0 = -\frac{a}{2r}$	$\text{21)} R_{303}^0 = -\frac{a \cdot \sin^2 \theta}{2r}$
$R_{110}^0 = -\frac{a}{r^2(r-a)}$	$R_{220}^0 = +\frac{a}{2r}$	$R_{330}^0 = +\frac{a \cdot \sin^2 \theta}{2r}$
$\text{1)} R_{001}^1 = \frac{a(r-a)}{r^4}$	$\text{16)} R_{212}^1 = -\frac{a}{2r}$	$\text{23)} R_{313}^1 = -\frac{a \cdot \sin^2 \theta}{2r}$
$R_{010}^1 = -\frac{a(r-a)}{r^4}$	$R_{221}^1 = +\frac{a}{2r}$	$R_{331}^1 = +\frac{a \cdot \sin^2 \theta}{2r}$
$\text{2)} R_{002}^2 = \frac{a(a-r)}{2r^4}$	$\text{10)} R_{112}^2 = \frac{a}{2r^2(r-a)}$	$\text{24)} R_{323}^2 = +\frac{a \cdot \sin^2 \theta}{r}$
$R_{020}^2 = -\frac{a(a-r)}{2r^4}$	$R_{121}^2 = -\frac{a}{2r^2(r-a)}$	$R_{332}^2 = -\frac{a \cdot \sin^2 \theta}{r}$
$\text{3)} R_{003}^3 = \frac{a(a-r)}{2r^4}$	$\text{11)} R_{113}^3 = \frac{a}{2r^2(r-a)}$	$\text{18)} R_{223}^3 = -\frac{a}{r}$
$R_{030}^3 = -\frac{a(a-r)}{2r^4}$	$R_{131}^3 = -\frac{a}{2r^2(r-a)}$	$R_{232}^3 = +\frac{a}{r}$

BASADA
DE ÍNDICES

Componentes NO NULAS del Tensor de Riemann
con todos los índices abajo

Utilizo la fórmula (16.4) de CRUZ : $R_{ijk} = g_{in} R^h_{\ jk}$

Como la métrica de Schwarzschild es diagonal, en el sumatorio de la fórmula sólo sobrevive el término en que $n=i \Rightarrow g_{ii}$

$$7) R_{0101} = g_{00} R^0_{101} = \frac{a-r}{r} \cdot \frac{a}{r^2(r-a)} \Rightarrow R_{0101} = -\frac{a}{r^3} = -R_{1010}$$

$$14) R_{0202} = g_{00} R^0_{202} = \frac{a-r}{r} \cdot \left(-\frac{a}{2r}\right) \Rightarrow R_{0202} = \frac{a(r-a)}{2r^2} = -R_{0220}$$

$$21) R_{0303} = g_{00} R^0_{303} = \frac{a-r}{r} \left(-\frac{a \cdot \operatorname{sen}^2 \theta}{2r}\right) \Rightarrow R_{0303} = \frac{a(r-a) \cdot \operatorname{sen}^2 \theta}{2r^2} = -R_{0330}$$

$$1) R_{1001} = g_{11} R^1_{001} = \frac{r}{r-a} \cdot \frac{a(r-a)}{r^4} \Rightarrow R_{1001} = \frac{a}{r^3} = -R_{1010}$$

$$16) R_{1212} = g_{11} R^1_{212} = \frac{r}{r-a} \cdot \left(-\frac{a}{2r}\right) \Rightarrow R_{1212} = \frac{a}{2(a-r)} = -R_{1221}$$

$$33) R_{1313} = g_{11} R^1_{313} = \frac{r}{r-a} \left(-\frac{a \cdot \operatorname{sen}^2 \theta}{2r}\right) \Rightarrow R_{1313} = \frac{a \cdot \operatorname{sen}^2 \theta}{2(a-r)} = -R_{1331}$$

$$2) R_{2002} = g_{22} R^2_{002} = r^2 \cdot \frac{a(a-r)}{2r^4} \Rightarrow R_{2002} = \frac{a(a-r)}{2r^2} = -R_{2020}$$

$$10) R_{2112} = g_{22} R^2_{112} = r^2 \cdot \frac{a}{2r^2(r-a)} \Rightarrow R_{2112} = \frac{a}{2(r-a)} = -R_{2121}$$

$$24) R_{2323} = g_{22} R^2_{323} = r^2 \cdot \frac{a \cdot \operatorname{sen}^2 \theta}{r} \Rightarrow R_{2323} = r \cdot a \cdot \operatorname{sen}^2 \theta = -R_{2332}$$

$$3) R_{3003} = g_{33} R^3_{003} = r^2 \cdot \operatorname{sen}^2 \theta \cdot \frac{a(a-r)}{2r^4} \Rightarrow R_{3003} = \frac{a(a-r) \operatorname{sen}^2 \theta}{2r^2} = -R_{3030}$$

$$11) R_{3113} = g_{33} R^3_{113} = r^2 \cdot \operatorname{sen}^2 \theta \cdot \frac{a}{2r^2(r-a)} \Rightarrow R_{3113} = \frac{a \cdot \operatorname{sen}^2 \theta}{2(r-a)} = -R_{3131}$$

$$18) R_{3223} = g_{33} R^3_{223} = r^2 \cdot \operatorname{sen}^2 \theta \cdot \left(-\frac{a}{r}\right) \Rightarrow R_{3223} = -r \cdot a \cdot \operatorname{sen}^2 \theta = -R_{3232}$$

Recordamos que $a = \frac{2GM}{c^2}$